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ON THE UNIMPORTANCE OF CONTINUOUS SPECTRUM IN THE PROBLEM OF STABILITY OF PLANE-PARALLEL FLOW OF PERFECT FLUID

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N. N. BOGDAT'EVA
(Moscow)

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It is shown that in the problem of stability of plane-parallel flows of perfect fluid only a discrete spectrum of eigenvalues exists. Previously this was established only in the case of monotonic velocity profile of the basic flow [1, 2]. Below a rigorous proof is given of this for any arbitrary profile.

The problem of stability of a plane-parallel flow of perfect fluid reduces to the Rayleigh equation whose solution is sought in the form of a wave $\varphi(z)\cdot\exp\{i\alpha(x-ct)\}$. The phase velocity c is the eigenvalue of this equation for conditions $\varphi(a)=\varphi(b)=0$ (see [3]), and the set of such eigenvalues constitutes the discrete spectrum. If stability is taken to mean the limitedness of any (not only wave) paerturbations in time, the question arises whether the continuous spectrum of the problem existing besides the discrete one can produce instability (as a rule, a discrete spectrum contains only a finite number of points). Existence of continuous spectrum becomes evident, if the Rayleigh equation is written thus:

$$(u+u''\Delta^{-1})\psi=c\psi, \quad \Delta=-d^2/dx^2+\alpha^2, \quad \psi=\Delta\varphi \tag{0.1}$$

where u is the velocity profile. The absolutely continuous operator $u''\Delta^{-1}$ cannot alter the continuous spectrum of the operator of multiplication by u, which means that such spectrum occupies the entire segment $[u_{\min}, u_{\max}]$.

It was shown in [1, 2] that only a discrete spectrum can generate instability, either exponential if c is not real, or power if there are multiple real eigenvalues. It was noted subsequently in [4] that only an outline of the proof is given in the short papers [1, 2], and an expanded proof by the same scheme is presented in it. However only the simplest case of a monotonic velocity profile u is fully investigated in all these papers. The same case was considered in [5], where it is shown that for a monotonic velocity profile the operator in the left-hand side of (0.1) is equivalent to the self-conjugate one for which spectral expansion exists. Hence the Cauchy problem can be solved by expanding in terms of the spectrum, and the realness of the continuous spectrum ensures the boundedness of the related part of the problem. The Laplace transformation used in [1, 2] is essentially a substitute for the general theorem on spectral expansion.

Here, the method of [1, 2] is extended to the case of a nonmonotonic velocity profile, and it is shown that such extension does not alter the fundamental result. (The discrete spectrum is not analyzed here, since virtually all works on stability deal with it. In particular, the necessary and sufficient condition for the stability of a monotonic velocity profile was obtained in [6] and derived again in [5]).

1. Let perturbation ψ be superposed on the basic flow at velocity $u(z)$ along the

Ox-axis. Linearizing the vortex equation, we obtain

$$\Delta\psi_t' + u\Delta\psi_x' - u''\psi_x' = 0$$

We seek a solution of the form $\exp[(i\alpha x)\,\varphi(z,t)]$ with conditions $\varphi(a,t) = \varphi(b,t) = 0$ and $\varphi(z,0) = \varphi_0(z)$.

Setting

$$\varphi^*(z,c) = \int_0^\infty e^{+i\alpha ct}\varphi(z,t)\,dt$$

we obtain

$$[u(z)-c]\left(\frac{d^2}{dz^2}-\alpha^2\right)\varphi^* - u''\varphi^* = f, \quad f = -i\,\frac{\varphi_0''-\varphi_0}{\alpha}$$

Let us express the solution in terms of Green's function

$$\varphi^*(z,c) = \int_a^b G^*(z,\zeta;c)\,f(\zeta)\,d\zeta \tag{1.1}$$

$$G^*(z,\zeta;c) = \begin{cases} \varphi_1(z,c)\,\varphi_2(\zeta,c)\,\{W(c)\,[u(\zeta)-c]\}^{-1}, & z<\zeta \\ \varphi_1(\zeta,c)\,\varphi_2(z,c)\,\{W(c)\,[u(\zeta)-c]\}^{-1}, & z\geqslant\zeta \end{cases}$$

$$W(c) = \varphi_1(b,c) = -\varphi_2(a,c)$$

where φ_1 and φ_2 are solutions of the homogeneous equation

$$(u-c)(\varphi''-\alpha^2\varphi) - u''\varphi = 0 \tag{1.2}$$

with conditions

$$\varphi_1(a,c) = 0,\ \ \varphi_1'(a,c) = 1,\ \ \varphi_2(b,c) = 0,\ \ \varphi_2'(b,c) = 1$$

The zeros of $W(c)$ are points of the discrete spectrum. For $c \to \infty$ function G^* decreases as c^{-1}, hence there exists function

$$G(z,\zeta;t) = \frac{\alpha}{2\pi}\int_{\gamma i-\infty}^{\gamma i+\infty} e^{-i\alpha ct}G^*(z,\zeta;c)\,dc \tag{1.3}$$

where γ is reasonably great, which yields solution

$$\varphi(z,t) = \int_a^b G(z,\zeta;t)\,f(\zeta)\,d\zeta \tag{1.4}$$

2. Let us estimate the behavior of $G(z,\zeta;t)$ for $t \to \infty$. For this we omit as far as possible the integration contour in (1.3). This requires the analytic continuation of G^* from the upper half-plane into some neighborhood of the real axis. We assume that $u(z)$ is analytically continued over the neighborhood of the real axis.

Lemma. Since $\varphi_1(z,c)$ and $\varphi_2(z,c)$ are functions of c, they continue analytically over the entire real axis, except at points $c = u(a)$, $u(b)$, $u(z)$ and $u(z^*)$, where z^* are extremal points and $u'(z^*) = 0$.

Proof of this appears in cited references, e.g., in [1].

Thus only points $c = u(a)$, $u(z)$ and $u(z^*)$ can be singular points of $\varphi_1(z,c)$, and for $\varphi_2(z,c)$ only points $u(b)$, $u(z)$ and $u(z^*)$. For the Wronskian $W(c)$ these are $u(a)$, $u(b)$ and $u(z^*)$. Function $G^*(z,\zeta;c)$ has singular points $c = u(a)$, $u(b)$, $u(z)$, $u(z^*)$ and $u(\zeta)$, and also zeros of function $W(c)$ different from $u(a)$,

$u(b)$ and $u(z^*)$. The zeros lead to the separation from the integral (1.3) of exponential terms which relate to the discrete spectrum. For multiple real zeros we obtain exponentially increasing terms. We consider $c = u(a)$ and $u(b)$ to be eigenvalues, if for these values solutions which are regular at the related end of the segment (and vanish there) vanish at its other end. The quantity $c = u(z^*)$ is considered to be an eigenvalue, if the unique solution regular for $z = z^*$ (and becomes a double zero there) vanishes at least at one end of the segment. In what follows we assume that $u(a)$, $u(b)$ and $u(z^*)$ are not eigenvalues. Proof of the following theorem is given in Sect. 3 below.

Theorem 1. If $u(a)$, $u(b)$ and $u(z^*)$ are not eigenvalues, there are no eigenvalues in their neighborhood.

Corollary. In this case there is only a finite number of points of the discrete spectrum in the region $\operatorname{Im} c > -\varepsilon$, where ε is a reasonably small positive number.

Theorem 2. If $u(a)$, $u(b)$ and $u(z^*)$ differ from each other and are not eigenvalues, and there are neither nonreal nor multiple real eigenvalues, the flow is stable.

3. Using the analyticity of the integrand in (1.1), we alter the form of the section of the integration path in the neighborhood of such ζ for which $u(\zeta)$ is an eigenvalue. The alteration is carried out according to the rule: down for $u'(\zeta) > 0$, up for $u'(\zeta) < 0$. We denote the altered segment by $[a, b]'$. Integration in formula (1.4) is also carried out along $[a, b]'$. Hence in what follows we always have $z \in [a, b]$, $\zeta \in [a, b]'$. Under these conditions function $u(\zeta)$ can never be closer to eigenvalues than the specified distance.

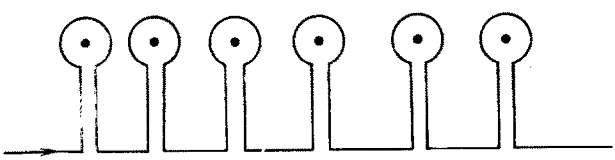

Fig. 1

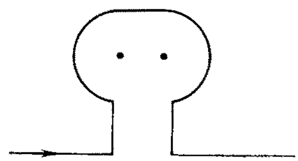

Fig. 2

To estimate $G(z, \zeta; t)$ for $t \to \infty$ we alter the contour in (1.3) by lowering it below the real axis, as shown in Fig. 1, by a fixed distance so that it bypasses the bifurcation points $u(a)$, $u(b)$, $u(z)$, $u(\zeta)$ and all $u(z^*)$. The poles of G^* which correspond to zeros of W yield residues which are exponents of t and additive to the integral. The contour is drawn so that there are no eigenvalues along it. The radii r of circles are chosen within C/t and $2C/t$, where C is a fixed positive number. If two or three circles intersect each other, an oval curve is substituted for these (Fig. 2). An additional condition will be imposed on the contour after Theorem 1 is proved.

Along the contour horizontal parts the function is continuous with respect to the set of variables z, ζ and c. Taking into consideration its attenuation when $c \to \infty$, we note that along this part of the contour it is uniformly bounded with respect to z, ζ and c. Let us now consider the behavior of $\varphi_1(z, c)$ in the neighborhood of singular points (the analysis in the case of $\varphi_2(z, c)$ is the same). We have to obtain an estimate which is valid for $c \in \Gamma_r$ (Fig. 1) and $z \in [a, b]$ or $[a, b]'$.

Lemma. If certain neighborhoods of $c = u(z^*)$ (z^* are extremal points) are fixed, then outside such neighborhoods the quantity $|\varphi_1(z, c)|$ is bounded by a single constant independent of radii r. If, however, c belongs to one of these neighborhoods, the following estimates hold: for z lying to the left of the point z^* neighborhood of radius $\sqrt{|c - u(z^*)|}$ we have

$$|\varphi_1(z,c)| < C(|z - z^*| + \sqrt{|c - u(z^*)|})^{-1} \quad (3.1)$$

and for z lying in that neighborhood or to the right of it

$$|\varphi_1(z,c)| < C(|z - z^*| + \sqrt{|c - u(z^*)|})^2 |c - u(z^*)|^{-3/2} \quad (3.2)$$

where C is a fixed constant. For $c \neq u(z^*)$ function $W(c)$ is continuously dependent on c. When $c \to u(z^*)$, then

$$W(c) \sim \text{const}\ |c - u(z^*)|^{-3/2}$$

Proof. Let, first, c lie outside some fixed neighborhood of $u(z^*)$ for all extremal points z^* but may approach $u(a)$ and $u(z)$. This case is simple and was the subject of a fairly detailed analysis (in for instance [4]) it is therefore, not considered here. The essential point here is that, although the solutions bifurcate at points z_c, where $u(z_c) = c$, they remain continuous. There are also no complications in the case of $u(z) = u(a)$, i.e. when the critical points simultaneously approach z and a. Since $W(c) = \varphi_1(b, c)$, it is also continuous for $c \to u(a)$ or $c \to u(b)$.

The difficulty arises when $c \to u(z^*)$ (hence this investigation). In this case two singular points z_c simultaneously approach point z^* and it is impossible to avoid these by any contour alteration, since the contour passes between the two points.

Let us assume that at the extremal point $u''(z^*) > 0$ (the degenerate case of $u''(z^*) = 0$ is not considered here). Then at $c^* = u(z^*)$ there is a reasonably small region O_c in which the neighborhood O_z of point z^* is mapped in a two-sheeted form by function

$$u(z) = u(z^*) + {}^1/_2\, u''(z^*)(z - z^*)^2 + \ldots$$

Two singular points z_c correspond to each $c \in O_c$. The image of O_z within which lies the image of the real axis in the form of the double contour, is shown in Fig. 3.

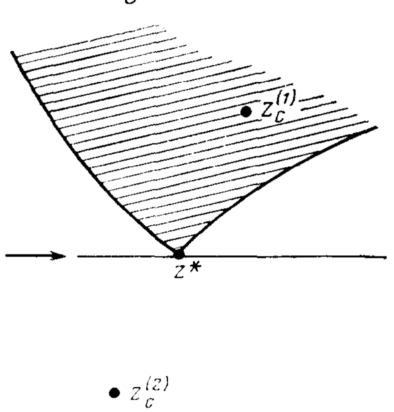

Fig. 3

Function $\varphi_1(z, c)$ is everywhere taken as its analytic continuation from the upper half-plane.

Two cases are to be distinguished here: $\mathrm{Re}\, c < u(z^*)$ and $\mathrm{Re}\, c \geqslant u(z^*)$.

a) $\mathrm{Re}\, c \leqslant u(z^*)$. One point z_c lies in curvilinear sector (cross-hatched in Fig. 3) whose boundaries diverge at angles $\pi/4$ and $3\pi/4$. The second point is approximately symmetric to the first. For $c \to u(z^*)$ points $z_c^{(1,2)}$ merge. Let us write $u(z) - c$ in the form $(z - z_1^1)(z - z_c^2)k$ and denote

$$z - z_c^2 = \xi, \quad z_c^2 - z_c^1 = \varepsilon, \quad (3.3)$$

$$[-u'' - \alpha^2(u - c)]\, k^{-1} = m(\xi, \varepsilon)$$

It is obvious that $|\varepsilon| \sim \sqrt{|c - u(z^*)|}$. The equation assumes the form

$$\xi(\xi-\varepsilon)\Phi''+m(\xi,\varepsilon)\Phi=0,\ \Phi(\xi,\varepsilon)=\varphi(z,c) \tag{3.4}$$

where $m(\xi,\varepsilon)$ is a smooth function of two variables, analytic with respect to ξ.

With respect to the asymptotic behavior of solutions of such equations we can prove the following. Let us consider equation

$$\xi^2\Phi''+m(\xi,0)\Phi=0 \tag{3.5}$$

whose limit occurs at $\varepsilon=0$. It has one solution $\Phi_0(\xi,0)$ which for $\xi=0$ tends to zero as ξ^{λ_1}, where λ_1 is the greatest root of the equation $\lambda(\lambda-1)+m(0,0)=0$, which is assumed to be real (it will be shown that in this case it is so). It is obvious that λ_1 is greater than $^1/_2$. Equation (3.4) has a solution which is analytic in zero and tends to vanish there as ξ. by suitable normalization the solution $\Phi_0(\xi,\varepsilon)$ tends uniformly to $\Phi_0(\xi,0)$ in the region of plane ξ shown in Fig. 3 outside the cross-hatched sector with fixed angle 2α. If in any region to the left of the neighborhood of the merging points $\xi=0$ and $\xi=\varepsilon$, any of the remaining solutions $\Phi(\xi,\varepsilon)$ of Eq. (3.4) tends to solution $\Phi(\xi,0)$ of Eq. (3.5), then to the right of that neighborhood such solution, being analytically continued along the contour which passes between the singular points $\xi=0$ and $\xi=\varepsilon$, represents the sum of the other branch of that function which bypasses the singular point $\xi=0$, and of the expression

$$Kw\Phi_0(\xi,0)\,\varepsilon^{1-2\lambda_1}[1+O(\varepsilon)]$$

where K is some absolute nonzero constant and w is the Wronskian of solutions $\Phi(\xi,0)$ and $\Phi_0(\xi,0)$. If $w\neq 0$, this term tends to infinity when $\varepsilon\to 0$ owing to $\lambda_1>{}^1/_2$. On the other hand, the first term, which is a function continued into the bypass of a singular point, is continuous for $\varepsilon\to 0$. For the neighborhood itself of merging singular points $\xi=0$ and $\xi=\varepsilon$ the following estimate is valid: in the $|\varepsilon|$-neighborhood of point $\xi=0$ and to the left of it

$$|\Phi(\xi,\varepsilon)|<C(|\xi|+|\varepsilon|)^{1-\lambda_1} \tag{3.6}$$

while to the right of that neighborhood

$$|\Phi(\xi,\varepsilon)|<C(|\xi|+|\varepsilon|)^{\lambda_1}|\varepsilon|^{1-2\lambda_1} \tag{3.7}$$

(in the neighborhood of $|\varepsilon|$ these estimates close in). Proof of this is given in [7]. Applying this to our case, we note that here $m(0,0)=-2$, as follows from (3.3), hence $\lambda_1=2$. Estimates (3.6), (3.7) and $|\varepsilon|\sim\sqrt{|c-u(z^*)|}$ yield (3.1) and (3.2).

The Wronskian of solutions $\varphi_1(z,c)$ and $\Phi(z,c)=\Phi_0(\xi,0)$ is nonzero, since otherwise a solution which for $c=c^*$ is regular at the extremal point z^* would satisfy the boundary condition for $z=a$, i.e. c^* would be an eigenvalue, which contradicts our assumption. It follows from this that for $c\to c^*$

$$W(c)\sim Kw\,\Phi_0(b,c^*)\,\varepsilon^{-3}[1+O(\varepsilon)] \tag{3.8}$$

But for the same reason $\Phi(b,c^*)\neq 0$, hence a solution which is regular at point z^* for $c=c^*$ cannot satisfy the boundary condition at the right-hand end. Thus in the region of $c=c^*$ function $W(c)$ tends to infinity.

b) $\operatorname{Re} c\geqslant u(z^*)$. The equation must be integrated along the altered contour.

Point $z_c^{(1)}$, analogous to point $z_c^{(1)}$ lying in Fig. 3 within the cross-hatched angle, now lies, as shown in Fig. 4, in the cross-hatched sector whose boundaries at point z^* diverge at angles $3\pi/4$ and $5\pi/4$. The integration contour is also shown in Fig. 4. All estimates remain valid. The lemma is proved.

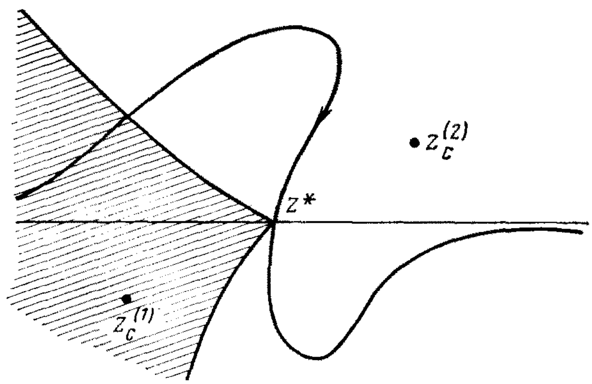

Fig. 4

The proof of Theorem 1 follows directly from the lemma. In fact, in the neighborhood of $c = u(z^*)$ function $W(c)$ cannot have zeros, since by virtue of (3.8) it tends to infinity, while, as shown earlier, at points $c = u(a)$ and $c = u(b)$ this function is continuous. If at these points it is nonzero (as assumed in the condition of the theorem), it is also nonzero in the indicated region.

4. Finally, let us prove Theorem 2. For this we estimate $G^*(z, \zeta; c)$ along the contour Γ_r. Since in the neighborhood of the real axis there is only a finite number of points of the discrete spectrum, radius r may be chosen so that the circles do not lie closer to eigenvalues than $r/4$. The contour horizontal part may be assumed to lie above all eigenvalues in the lower half-plane. The estimate

$$|G^*(z, \zeta; c)| < \text{const}/r$$

is then valid.

In fact, outside the fixed neighborhood of extremal points the numerator in formula (1.1) for Green's function is limited. The multipliers $u(\zeta) - c$ and $W(c)$ are not smaller in modulo than the constant multiplied by r, and both cannot be simultaneously small, since for $\zeta \in [a; b]'$ the expression $u(\zeta)$ nowhere approaches the eigenvalue.

Let us now consider the values of c in the neighborhood of $u(z^*)$.

Let at the beginning $z \leqslant \zeta \leqslant z^*$. Then $|z - z^*| \geqslant |\zeta - z^*|$ and by the last lemma we have

$$|\varphi_1(z, c)\varphi_2(\zeta, c)| < \text{const}\cdot(|z - z^*| + \sqrt{|c - u(z^*)|})^{-1} \times$$
$$(|\zeta - z^*| + \sqrt{|c - u(z^*)|})^2(|c - u(z^*)|)^{-3} <$$
$$\text{const}\cdot(|\zeta - z^*| + \sqrt{|c - u(z^*)|})(\sqrt{|c - u(z^*)|})^{-3} <$$
$$\text{const}\cdot|c - u(z^*)|^{-3/2}$$

(estimates by the last lemma are obviously applicable to φ_2, if words "to the left of" are substituted for "to the right of" and vice versa). Taking into account that $W(c) \sim$ const $|c - u(z^*)|^{-3/2}$, we obtain the required estimate for G^*.

For $z \leqslant z^* \leqslant \zeta$ we obtain the even stronger estimate

$$|\varphi_1(z, c)\varphi_2(\zeta, c)| \leqslant \text{const}\cdot(|z - z^*| + \sqrt{|c - u(z^*)|})^{-1} \times$$
$$(|\zeta - z^*| + \sqrt{|c - u(z^*)|})^{-1} < \text{const}\cdot|c - u(z^*)|^{-1}$$

The remaining cases of symmetrical disposition of z, z^* and ζ reduce to the case considered above.

Let us now recall the transformation formula

$$G(z, \zeta; t) = \frac{\alpha}{2\pi}\int_{\Gamma_r} e^{-i\alpha ct} G^*(z, \zeta; c)\, dc + \ldots$$

where the dots denote the sum of residues over the discrete spectrum. For $t \to \infty$ and owing to $\exp(-i\alpha ct)$ and the boundedness of $|G^*|$, the integral Γ_r along the horizontal part of the contour tends exponentially to zero. Along the circles values of $\exp(-i\alpha ct)$ are limited, since r is of the order of t^{-1}. The length of circles is of the order of r, and $|G^*| < Cr^{-1}$. Hence the integrals along the circles are bounded for $t \to \infty$.

Integrals over vertical segments

$$\left|\int e^{-i\alpha ct} G^* dc\right| < C\int_0^\infty \exp(\alpha \operatorname{Im} ct)\, td\,|\operatorname{Im} c| < \text{const}$$

are bounded for the same reason.

Theroem 2 is proved.

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